Chapter 4

INVERSE KINEMATICS

In the previous chapter we showed how to determine the end-effector position and orientation in terms of the joint variables. This chapter is concerned with the inverse problem of finding the joint variables in terms of the end-effector position and orientation. This is the problem of **inverse kinematics**, and it is, in general, more difficult than the forward kinematics problem.

In this chapter, we begin by formulating the general inverse kinematics problem. Following this, we describe the principle of kinematic decoupling and how it can be used to simplify the inverse kinematics of most modern manipulators. Using kinematic decoupling, we can consider the position and orientation problems independently. We describe a geometric approach for solving the positioning problem, while we exploit the Euler angle parameterization to solve the orientation problem.

4.1 The General Inverse Kinematics Problem

The general problem of inverse kinematics can be stated as follows. Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & O \\ 0 & 1 \end{bmatrix} \in SE(3) \tag{4.1}$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$T_n^0(q_1, \ldots, q_n) = H \tag{4.2}$$

where

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n). \tag{4.3}$$

Here, H represents the desired position and orientation of the end-effector, and our task is to find the values for the joint variables $q_1, \ldots, q_n$ so that $T_n^0(q_1, \ldots, q_n) = H$.

Equation (4.2) results in twelve nonlinear equations in n unknown variables, which can be written as

$$T_{ij}(q_1, \ldots, q_n) = h_{ij}, \qquad i = 1, 2, 3, \quad j = 1, \ldots, 4 \tag{4.4}$$

where T_{ij}, h_{ij} refer to the twelve nontrivial entries of T_n^0 and H, respectively. (Since the bottom row of both T_n^0 and H are (0,0,0,1), four of the sixteen equations represented by (4.2) are trivial.)

Example 4.1

Recall the Stanford manipulator of Example 3.3.5. Suppose that the desired position and orientation of the final frame are given by

$$H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & O_x \\ r_{21} & r_{22} & r_{23} & O_y \\ r_{31} & r_{32} & r_{33} & O_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations (cf. (3.43) and (3.44)):

$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= r_{11} \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= r_{21} \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5s_6 &= r_{31} \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= r_{12} \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= r_{22} \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= r_{32} \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= r_{13} \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= r_{23} \\
-s_2c_4s_5 + c_2c_5 &= r_{33} \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= O_x \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= O_y \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= O_z.
\end{aligned}$$

◇

The equations in the preceding example are, of course, much too difficult to solve directly in closed form. This is the case for most robot arms. Therefore, we need to develop efficient and systematic techniques that exploit the particular kinematic structure of the manipulator. Whereas the forward kinematics problem always has a unique solution that can be obtained simply by evaluating the forward equations, the inverse kinematics problem may or may not have a solution. Even if a solution exists, it may or may not be unique. Furthermore, because these forward kinematic equations are in general complicated nonlinear functions of the joint variables, the solutions may be difficult to obtain even when they exist.

In solving the inverse kinematics problem we are most interested in finding a closed form solution of the equations rather than a numerical solution. Finding a closed form solution means finding an explicit relationship:

$$q_k = f_k(h_{11}, \ldots, h_{34}), \qquad k = 1, \ldots, n. \tag{4.6}$$

Closed form solutions are preferable for two reasons. First, in certain applications, such as tracking a welding seam whose location is provided by a vision system, the inverse kinematic equations must be solved at a rapid rate, say every 20 milliseconds, and having closed form expressions rather than an iterative search is a practical necessity. Second, the kinematic equations in general have multiple solutions. Having closed form solutions allows one to develop rules for choosing a particular solution among several.

The practical question of the existence of solutions to the inverse kinematics problem depends on engineering as well as mathematical considerations. For example, the motion of the revolute joints may be restricted to less than a full 360 degrees of rotation so that not all mathematical solutions of the kinematic equations will correspond to physically realizable configurations of the manipulator. We will assume that the given position and orientation is such that at least one solution of (4.2) exists. Once a solution to the mathematical equations is identified, it must be further checked to see whether or not it satisfies all constraints on the ranges of possible joint motions. For our purposes here we henceforth assume that the given homogeneous matrix H in (4.2) corresponds to a configuration within the manipulator's workspace with an attainable orientation. This then guarantees that the mathematical solutions obtained correspond to achievable configurations.

4.2 Kinematic Decoupling

Although the general problem of inverse kinematics is quite difficult, it turns out that for manipulators having six joints, with the last three joints intersecting at a point (such as the Stanford Manipulator above), it is possible to decouple the inverse kinematics problem into two simpler problems, known respectively, as **inverse position kinematics**, and **inverse orientation kinematics**. To put it another way, for a six-DOF manipulator with a spherical wrist, the inverse kinematics problem may be separated into two simpler problems, namely first finding the position of the intersection of the wrist axes, hereafter called the **wrist center**, and then finding the orientation of the wrist.

For concreteness let us suppose that there are exactly six degrees-of-freedom and that the last three joint axes intersect at a point O_c. We express (4.2) as two sets of equations representing the rotational and positional equations

$$R_6^0(q_1, \ldots, q_6) = R \tag{4.7}$$
$$O_6^0(q_1, \ldots, q_6) = O \tag{4.8}$$

where O and R are the desired position and orientation of the tool frame, expressed with respect to the world coordinate system. Thus, we are given O and R, and the inverse kinematics problem is to solve for $q_1, \ldots, q_6$.

The assumption of a spherical wrist means that the axes z_3, z_4, and z_5 intersect at O_c and hence the origins O_4 and O_5 assigned by the DH-convention will always be at the wrist center O_c. Often O_3 will also be at O_c, but this is not necessary for our subsequent development. The important point of this assumption for the inverse kinematics is that motion of the final three links about these axes will not change the position of O_c, and thus, the position of the wrist center is thus a function of only the first three joint variables.

The origin of the tool frame (whose desired coordinates are given by O) is simply obtained by a translation of distance d_6 along z_5 from O_c (see Table 3.3). In our case, z_5 and z_6 are the same axis, and the third column of R expresses the direction of z_6 with respect to the base frame. Therefore, we have

$$O = O_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.9}$$

Thus in order to have the end-effector of the robot at the point with coordinates given by O and with the orientation of the end-effector given by $R = (r_{ij})$, it is necessary and sufficient that the wrist center O_c have coordinates given by

$$O_c^0 \;=\; O - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.10}$$

and that the orientation of the frame $o_6x_6y_6z_6$ with respect to the base be given by R. If the components of the end-effector position O are denoted O_x, O_y, O_z and the components of the wrist center O_c^0 are denoted x_c, y_c, z_c then (4.10) gives the relationship

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \;=\; \begin{bmatrix} O_x - d_6 r_{13} \\ O_y - d_6 r_{23} \\ O_z - d_6 r_{33} \end{bmatrix}. \tag{4.11}$$

Using Equation (4.11) we may find the values of the first three joint variables. This determines the orientation transformation R_3^0 which depends only on these first three joint variables. We can now determine the orientation of the end-effector relative to the frame $o_3x_3y_3z_3$ from the expression

$$R \;=\; R_3^0 R_6^3 \tag{4.12}$$

as

$$R_6^3 \;=\; (R_3^0)^{-1} R = (R_3^0)^T R. \tag{4.13}$$

As we shall see in Section 4.4, the final three joint angles can then be found as a set of Euler angles corresponding to R_6^3. Note that the right hand side of (4.13) is completely known since R is given and R_3^0 can be calculated once the first three joint variables are known. The idea of kinematic decoupling is illustrated in Figure 4.1.

4.2.1 Summary

For this class of manipulators the determination of the inverse kinematics can be summarized by the following algorithm.

Step 1: Find q_1, q_2, q_3 such that the wrist center O_c has coordinates given by

$$O_c^0 \;=\; O - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.14}$$

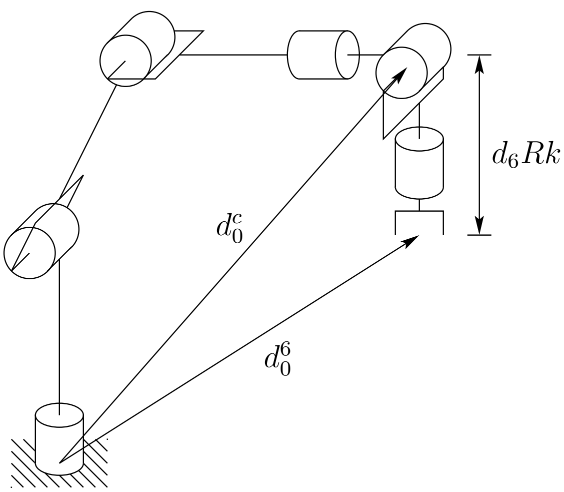

Figure 4.1: Kinematic decoupling.

Step 2: Using the joint variables determined in Step 1, evaluate R_3^0.

Step 3: Find a set of Euler angles corresponding to the rotation matrix

$$R_6^3 = (R_3^0)^{-1}R = (R_3^0)^T R. \tag{4.15}$$

4.3 Inverse Position: A Geometric Approach

For the common kinematic arrangements that we consider, we can use a geometric approach to find the variables, q_1, q_2, q_3 corresponding to O_c^0 given by (4.10). We restrict our treatment to the geometric approach for two reasons. First, as we have said, most present manipulator designs are kinematically simple, usually consisting of one of the five basic configurations of Chapter 1 with a spherical wrist. Indeed, it is partly due to the difficulty of the general inverse kinematics problem that manipulator designs have evolved to their present state. Second, there are few techniques that can handle the general inverse kinematics problem for arbitrary configurations. Since the reader is most likely to encounter robot configurations of the type considered here, the added difficulty involved in treating the general case seems unjustified. The reader is directed to the references at the end of the chapter for treatment of the general case.

In general the complexity of the inverse kinematics problem increases with the number of nonzero link parameters. For most manipulators, many of the a_i, d_i are zero, the α_i are 0 or $\pm\pi/2$, etc. In these cases especially, a geometric approach is the simplest and most natural. We will illustrate this with several important examples.

4.3.1 Articulated Configuration

Consider the elbow manipulator shown in Figure 4.2, with the components of O_c^0 denoted by x_c, y_c, z_c. We project O_c onto the $x_0 - y_0$ plane as shown in Figure 4.3.

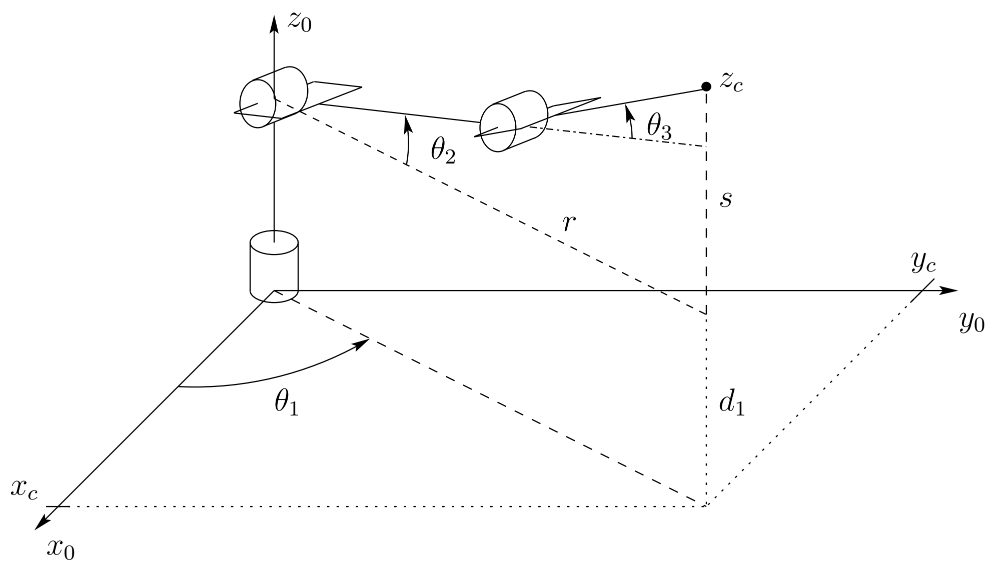

Figure 4.2: Elbow manipulator.

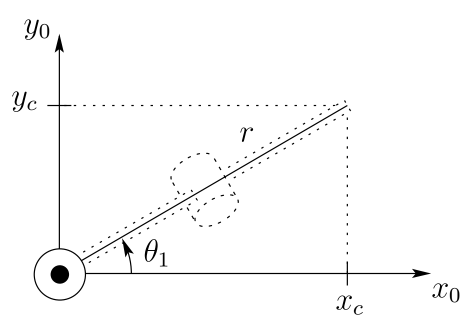

Figure 4.3: Projection of the wrist center onto $x_0 - y_0$ plane.

We see from this projection that

$$\theta_1 = A\tan(x_c, y_c), \tag{4.16}$$

in which $A\tan(x,y)$ denotes the two argument arctangent function. $A\tan(x,y)$ is defined for all $(x,y) \neq (0,0)$ and equals the unique angle θ such that

$$\cos\theta = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}, \qquad \sin\theta = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}. \tag{4.17}$$

For example, $A\tan(1,-1) = -\frac{\pi}{4}$, while $A\tan(-1,1) = +\frac{3\pi}{4}$.

Note that a second valid solution for θ_1 is

$$\theta_1 = \pi + A\tan(x_c, y_c). \tag{4.18}$$

Of course this will, in turn, lead to different solutions for θ_2 and θ_3, as we will see below.

These solutions for θ_1, are valid unless $x_c = y_c = 0$. In this case (4.16) is undefined and the manipulator is in a singular configuration, shown in Figure 4.4. In this position the wrist center O_c intersects z_0; hence any

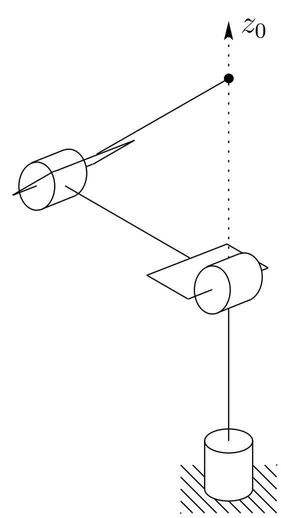

Figure 4.4: Singular configuration.

value of θ_1 leaves O_c fixed. There are thus infinitely many solutions for θ_1 when O_c intersects z_0.

If there is an offset $d \neq 0$ as shown in Figure 4.5 then the wrist center cannot intersect z_0. In this case, depending on how the DH parameters

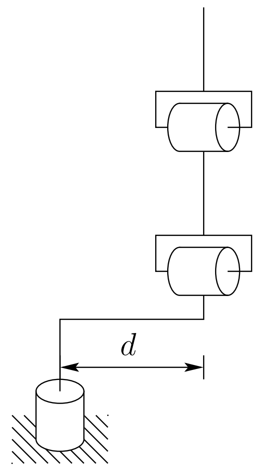

Figure 4.5: Elbow manipulator with shoulder offset.

have been assigned, we will have $d_2 = d$ or $d_3 = d$. In this case, there will, in general, be only two solutions for θ_1. These correspond to the so-called **left arm** and **right arm** configurations as shown in Figures 4.6 and 4.7. Figure 4.6 shows the left arm configuration. From this figure, we see geometrically that

$$\theta_1 = \phi - \alpha \tag{4.19}$$

where

$$\begin{aligned} \phi &= A\tan(x_c, y_c) && (4.20) \\ \alpha &= A\tan\left(\sqrt{r^2 - d^2}, d\right) && (4.21) \\ &= A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right). \end{aligned}$$

The second solution, given by the right arm configuration shown in Figure 4.7 is given by

$$\theta_1 = A\tan(x_c, y_c) + A\tan\left(-\sqrt{r^2 - d^2}, -d\right). \tag{4.22}$$

To see this, note that

$$\begin{aligned} \theta_1 &= \alpha + \beta && (4.23) \\ \alpha &= A\tan(x_c, y_c) && (4.24) \end{aligned}$$

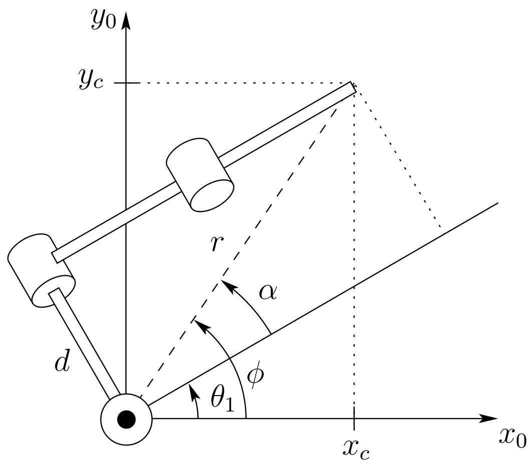

Figure 4.6: Left arm configuration.

$$\beta = \gamma + \pi \tag{4.25}$$
$$\gamma = A\tan(\sqrt{r^2 - d^2}, d) \tag{4.26}$$
$$\tag{4.27}$$

which together imply that

$$\beta = A\tan\left(-\sqrt{r^2 - d^2}, -d\right) \tag{4.28}$$

since $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$.

To find the angles θ_2, θ_3 for the elbow manipulator, given θ_1, we consider the plane formed by the second and third links as shown in Figure 4.8. Since the motion of links two and three is planar, the solution is analogous to that of the two-link manipulator of Chapter 1. As in our previous derivation (cf. (1.8) and (1.9)) we can apply the law of cosines to obtain

$$\begin{aligned} \cos\theta_3 &= \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3} \qquad (4.29) \\ &= \frac{x_c^2 + y_c^2 - d^2 + z_c^2 - a_2^2 - a_3^2}{2a_2a_3} := D, \end{aligned}$$

since $r^2 = x_c^2 + y_c^2 - d^2$ and $s = z_c$. Hence, θ_3 is given by

$$\theta_3 = A\tan\left(D, \pm\sqrt{1 - D^2}\right). \tag{4.30}$$

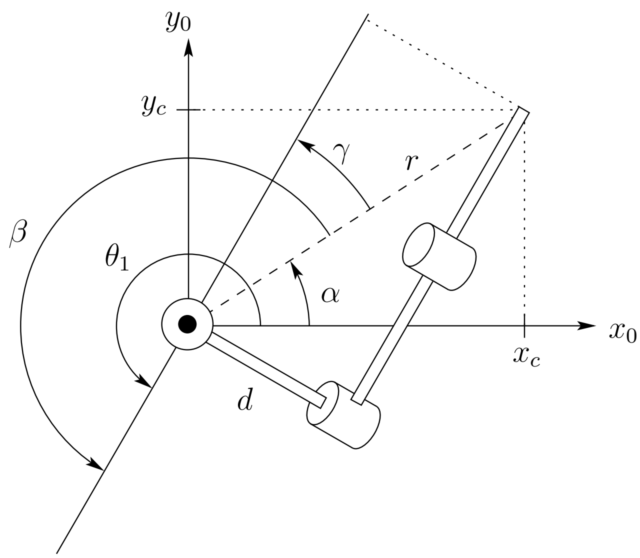

Figure 4.7: Right arm configuration.

Similarly θ_2 is given as

$$\begin{aligned}\theta_2 &= A\tan(r, s) - A\tan(a_2 + a_3c_3, a_3s_3) \\ &= A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c\right) - A\tan(a_2 + a_3c_3, a_3s_3).\end{aligned} \tag{4.31}$$

The two solutions for θ_3 correspond to the elbow-up position and elbow-down position, respectively.

An example of an elbow manipulator with offsets is the PUMA shown in Figure 4.9. There are four solutions to the inverse position kinematics as shown. These correspond to the situations left arm-elbow up, left arm–elbow down, right arm–elbow up and right arm–elbow down. We will see that there are two solutions for the wrist orientation thus giving a total of eight solutions of the inverse kinematics for the PUMA manipulator.

4.3.2 Spherical Configuration

We next solve the inverse position kinematics for a three degree of freedom spherical manipulator shown in Figure 4.10. As in the case of the elbow manipulator the first joint variable is the base rotation and a solution is given as

$$\theta_1 = A\tan(x_c, y_c) \tag{4.32}$$

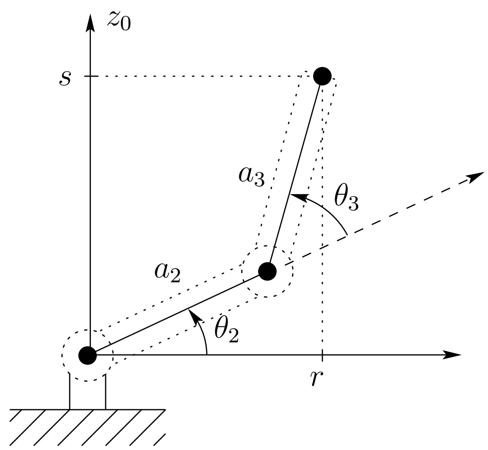

Figure 4.8: Projecting onto the plane formed by links 2 and 3.

provided x_c and y_c are not both zero. If both x_c and y_c are zero, the configuration is singular as before and θ_1 may take on any value.

The angle θ_2 is given from Figure 4.10 as

$$\theta_2 = A\tan(r, s) + \frac{\pi}{2} \tag{4.33}$$

where $r^2 = x_c^2 + y_c^2$, $s = z_c - d_1$. As in the case of the elbow manipulator a second solution for θ_1 is given by

$$\theta_1 = \pi + A\tan(x_c, y_c); \tag{4.34}$$

The linear distance d_3 is found as

$$d_3 = \sqrt{r^2 + s^2} = \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}. \tag{4.35}$$

The negative square root solution for d_3 is disregarded and thus in this case we obtain two solutions to the inverse position kinematics as long as the wrist center does not intersect z_0. If there is an offset then there will be left and right arm configurations as in the case of the elbow manipulator (Problem 4-12).

4.4 Inverse Orientation

In the previous section we used a geometric approach to solve the inverse position problem. This gives the values of the first three joint variables

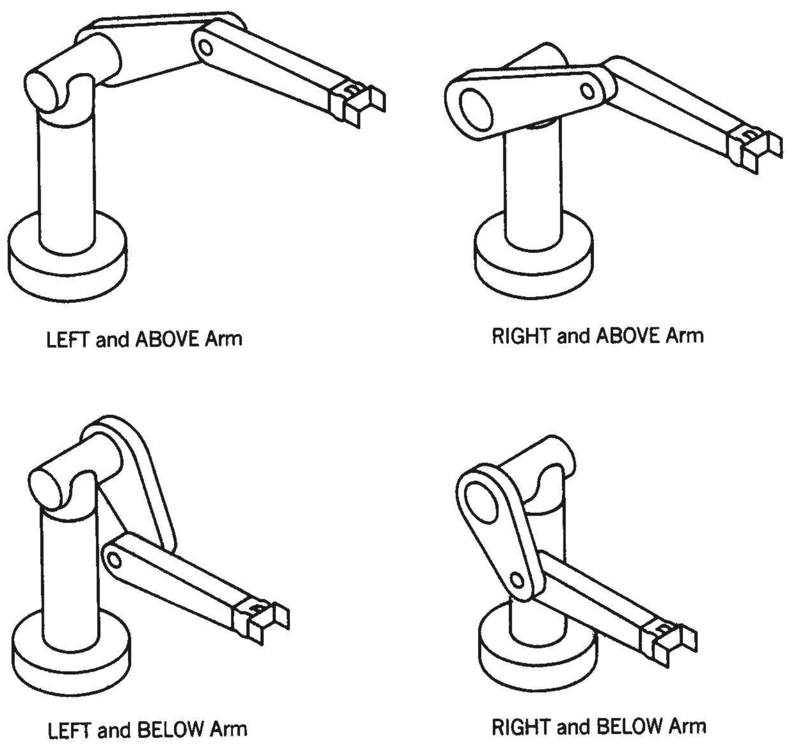

Figure 4.9: Four solutions of the inverse position kinematics for the PUMA manipulator.

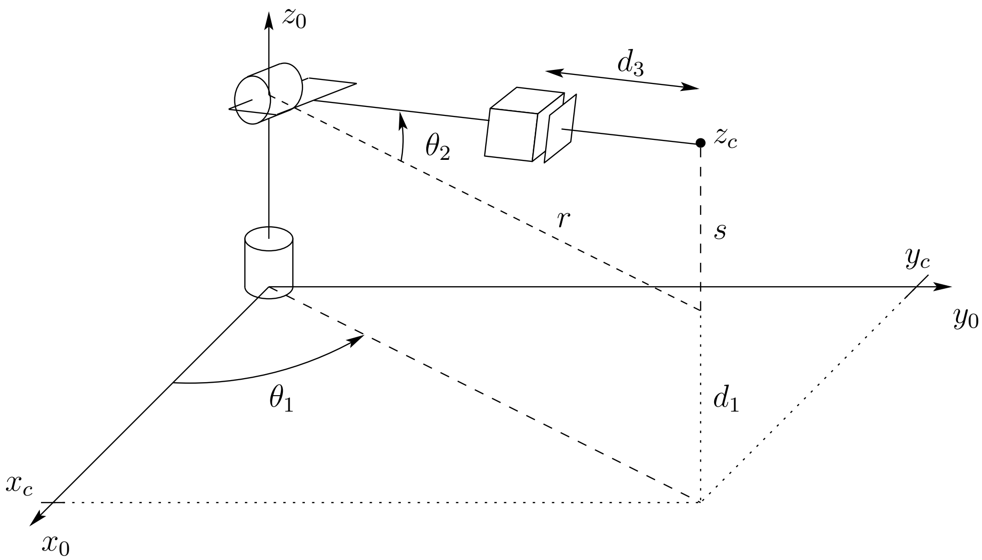

Figure 4.10: Spherical manipulator.

corresponding to a given position of the wrist origin. The inverse orientation problem is now one of finding the values of the final three joint variables corresponding to a given orientation with respect to the frame $o_3x_3y_3z_3$. For a spherical wrist, this can be interpreted as the problem of finding a set of Euler angles corresponding to a given rotation matrix R. Recall that equation (3.32) shows that the rotation matrix obtained for the spherical wrist has the same form as the rotation matrix for the Euler transformation, given in (2.52). Therefore, we can use the method developed in Section 2.5.1 to solve for the three joint angles of the spherical wrist. In particular, we solve for the three Euler angles, ϕ, θ, ψ, using Equations (2.54) – (2.59), and then use the mapping

$$\begin{aligned}\theta_4 &= \phi,\\ \theta_5 &= \theta,\\ \theta_6 &= \psi.\end{aligned}$$

Example 4.2 Articulated Manipulator with Spherical Wrist

The DH parameters for the frame assignment shown in Figure 4.2 are summarized in Table 4.1. Multiplying the corresponding A_i matrices gives

Table 4.1: Link parameters for the articulated manipulator of Figure 4.2.

Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*

* variable

the matrix R_3^0 *for the articulated or elbow manipulator as*

$$R_3^0 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}. \tag{4.36}$$

The matrix $R_6^3 = A_4A_5A_6$ *is given as*

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}. \tag{4.37}$$

The equation to be solved now for the final three variables is therefore

$$R_6^3 = (R_3^0)^T R \tag{4.38}$$

and the Euler angle solution can be applied to this equation. For example, the three equations given by the third column in the above matrix equation are given by

$$c_4s_5 = c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33} \tag{4.39}$$

$$s_4s_5 = -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33} \tag{4.40}$$

$$c_5 = s_1r_{13} - c_1r_{23}. \tag{4.41}$$

Hence, if not both of the expressions (4.39), (4.40) are zero, then we obtain θ_5 *from (2.54) and (2.55) as*

$$\theta_5 = A\tan\left(s_1r_{13} - c_1r_{23}, \pm\sqrt{1 - (s_1r_{13} - c_1r_{23})^2}\right). \tag{4.42}$$

If the positive square root is chosen in (4.42), then θ_4 *and* θ_6 *are given by (2.56) and (2.57), respectively, as*

$$\begin{aligned} \theta_4 &= A\tan(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33}, \\ &\qquad -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \qquad (4.43) \\ \theta_6 &= A\tan(-s_1r_{11} + c_1r_{21}, s_1r_{12} - c_1r_{22}). \qquad (4.44) \end{aligned}$$

The other solutions are obtained analogously. If $s_5 = 0$, then joint axes z_3 and z_5 are collinear. This is a singular configuration and only the sum $\theta_4 + \theta_6$ can be determined. One solution is to choose θ_4 arbitrarily and then determine θ_6 using (2.62) or (2.64).
◇

Example 4.3 Summary of Elbow Manipulator Solution

To summarize the preceding development we write down one solution to the inverse kinematics of the six degree-of-freedom elbow manipulator shown in Figure 4.2 which has no joint offsets and a spherical wrist.

Given

$$O = \begin{bmatrix} O_x \\ O_y \\ O_z \end{bmatrix}; \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{4.45}$$

then with

$$x_c = O_x - d_6 r_{13} \tag{4.46}$$
$$y_c = O_y - d_6 r_{23} \tag{4.47}$$
$$z_c = O_z - d_6 r_{33} \tag{4.48}$$

a set of D-H joint variables is given by

$$\theta_1 = A\tan(x_c, y_c) \tag{4.49}$$
$$\theta_2 = A\tan\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c\right) - A\tan(a_2 + a_3 c_3, a_3 s_3) \tag{4.50}$$
$$\theta_3 = A\tan\left(D, \pm\sqrt{1 - D^2}\right), \quad \textit{where } D = \frac{x_c^2 + y_c^2 - d^2 + z_c^2 - a_2^2 - a_3^2}{2a_2 a_3} \tag{4.51}$$
$$\theta_4 = A\tan(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}) \tag{4.52}$$
$$\theta_5 = A\tan\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}\right). \tag{4.53}$$
$$\theta_6 = A\tan(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22}). \tag{4.54}$$

The other possible solutions are left as an exercise (Problem 4-11). ◇

Example 4.4 SCARA Manipulator

As another example, we consider the SCARA manipulator whose forward kinematics is defined by T_4^0 from (3.49). The inverse kinematics is then given as the set of solutions of the equation

$$\left[\begin{array}{cccc} c_{12}c_4 + s_{12}s_4 & s_{12}c_4 - c_{12}s_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -c_{12}c_4 - s_{12}s_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc} R & O \\ 0 & 1 \end{array}\right]. \tag{4.55}$$

We first note that, since the SCARA has only four degrees-of-freedom, not every possible H from $SE(3)$ allows a solution of (4.55). In fact we can easily see that there is no solution of (4.55) unless R is of the form

$$R = \left[\begin{array}{ccc} c_\alpha & s_\alpha & 0 \\ s_\alpha & -c_\alpha & 0 \\ 0 & 0 & -1 \end{array}\right] \tag{4.56}$$

and if this is the case, the sum $\theta_1 + \theta_2 - \theta_4$ is determined by

$$\theta_1 + \theta_2 - \theta_4 = \alpha = A\tan(r_{11}, r_{12}). \tag{4.57}$$

Projecting the manipulator configuration onto the $x_0 - y_0$ plane immediately yields the situation of Figure 4.11.
We see from this that

$$\theta_2 = A\tan\left(c_2, \pm\sqrt{1 - c_2}\right) \tag{4.58}$$

where

$$c_2 = \frac{O_x^2 + O_y^2 - a_1^2 - a_2^2}{2a_1a_2} \tag{4.59}$$

$$\theta_1 = A\tan(O_x, O_y) - A\tan(a_1 + a_2c_2, a_2s_2). \tag{4.60}$$

We may then determine θ_4 from (4.57) as

$$\begin{aligned} \theta_4 &= \theta_1 + \theta_2 - \alpha \qquad (4.61) \\ &= \theta_1 + \theta_2 - A\tan(r_{11}, r_{12}). \end{aligned}$$

Finally d_3 is given as

$$d_3 = O_z + d_4. \tag{4.62}$$

◇

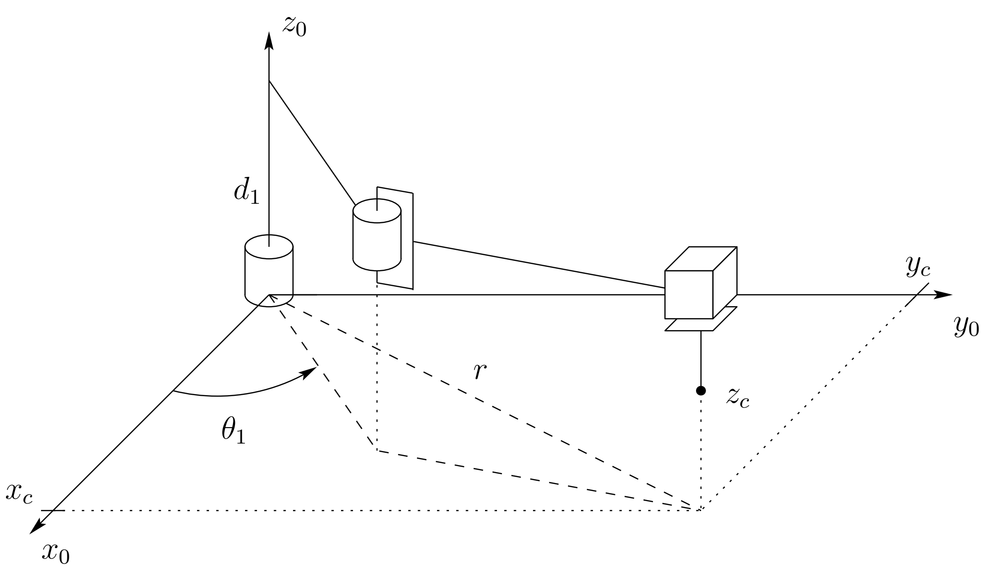

Figure 4.11: SCARA manipulator.

4.5 Problems

1. Given a desired position of the end-effector, how many solutions are there to the inverse kinematics of the three-link planar arm shown in Figure 4.12? If the orientation of the end-effector is also specified, how

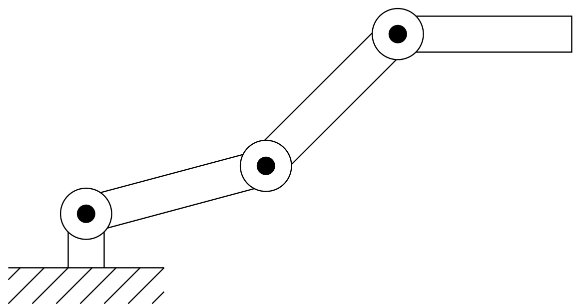

Figure 4.12: Three-link planar robot with revolute joints.

many solutions are there? Use the geometric approach to find them.

2. Repeat Problem 4-1 for the three-link planar arm with prismatic joint of Figure 4.13.

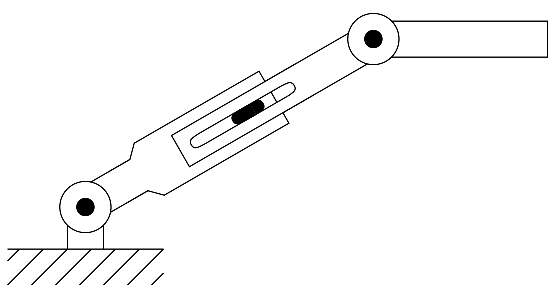

Figure 4.13: Three-link planar robot with prismatic joint.

3. Solve the inverse position kinematics for the cylindrical manipulator of Figure 4.14.

4. Solve the inverse position kinematics for the cartesian manipulator of Figure 4.15.

5. Add a spherical wrist to the three-link cylindrical arm of Problem 4-3 and write the complete inverse kinematics solution.

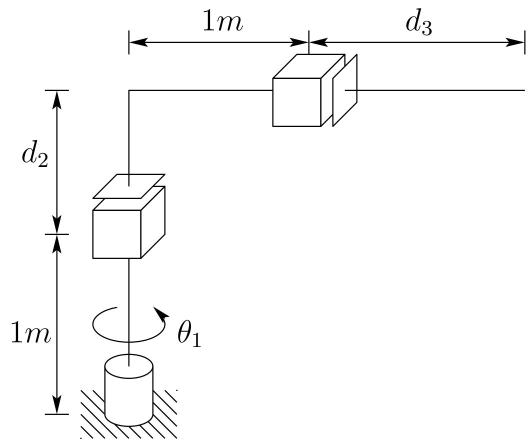

Figure 4.14: Cylindrical configuration.

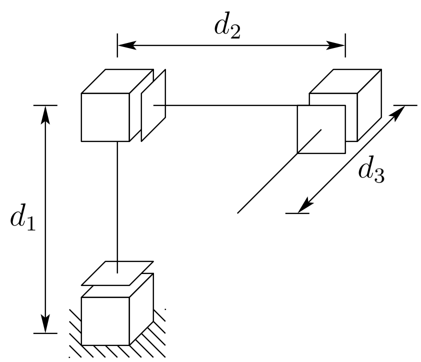

Figure 4.15: Cartesian configuration.

6. Repeat Problem 4-5 for the cartesian manipulator of Problem 4-4.

7. Write a computer program to compute the inverse kinematic equations for the elbow manipulator using Equations (4.49)-(4.54). Include procedures for identifying singular configurations and choosing a particular solution when the configuration is singular. Test your routine for various special cases, including singular configurations.

8. The Stanford manipulator of Example 3.3.5 has a spherical wrist. Therefore, given a desired position O and orientation R of the end-effector,

   a) Compute the desired coordinates of the wrist center O_c^0.

b) Solve the inverse position kinematics, that is, find values of the first three joint variables that will place the wrist center at O_c. Is the solution unique? How many solutions did you find?

c) Compute the rotation matrix R_3^0. Solve the inverse orientation problem for this manipulator by finding a set of Euler angles corresponding to R_6^3 given by (4.37).

9. Repeat Problem 4-8 for the PUMA 260 manipulator of Problem 3-9, which also has a spherical wrist. How many total solutions did you find?

10. Solve the inverse position kinematics for the Rhino robot.

11. Find all other solutions to the inverse kinematics of the elbow manipulator of Example 4.4.1.

12. Modify the solutions θ_1 and θ_2 for the spherical manipulator given by Equations (4.32) and (4.33) in the case of a shoulder offset.